Proof format:

1) Show, for two-node path, that $f_1 = f_2$ optimizes D.

2) Show that, starting from n-node path and appending a node $f_n = f_1$, given that $\forall_{i=1}^{n-1} f_i = f_{i+1}$.

1) $D = (g_1 h_1 + p_1) + (g_2 h_2 + p_2)$

$h_1 h_2 = H$

$h_2 = H/h_1$

$D = (g_1 h_1 + p_1) + (g_2 H/h_1 + p_2)$

$\partial D/\partial h_1 = (g_1) + (-g_2 H/h_1^2)$

$0 = g_1 - g_2 H/h_1^2$

$h_2 = H/h_1$

$0 = g_1 - g_2 h_2/h_1$

$g_1 = g_2 h_2/h_1$

$g_1 h_1 = g_2 h_2$

$f_i = g_i h_i$

$f_1 = f_2$ $\square$

2) $D = \left(\sum_{i=1}^{n}(g_i h_i + p_i)\right) + (g' h' + p')$

Given: $\forall_{i=1}^{n-1} f_i = f_{i+1}$

$f_i = g_i h_i$

$D = \left(g_1 h_1 n + \sum_{i=1}^{n} p_i\right) + (g' h' + p')$

$\forall_{i=1}^{n} h_1 g_1 = h_i g_i$

$h_i = h_1 g_1 / g_i$

$H_n = \prod_{i=1}^{n} h_i = \prod_{i=1}^{n} h_1 g_1 / g_i = h_1^n g_1^n \cdot \prod_{i=1}^{n} 1/g_i = h_1^n g_1^n / G_n$

$H = H_n \cdot h'$

$H = \dfrac{h_1^n g_1^n h'}{G_n}$

$h' = \dfrac{H G_n}{h_1^n g_1^n}$

$D = \left(g_1 h_1 n + \sum_{i=1}^{n} p_i\right) + \left(\dfrac{g' H G_n}{h_1^n g_1^n} + p'\right)$

$\partial D / \partial h_1 = g_1 n - \dfrac{g' H G_n n}{g_1^n h_1^{n+1}}$

$0 = g_1 n - \dfrac{g' H G_n \cdot n}{g_1^n h_1^{n+1}}$

$g_1 n = \dfrac{g' H G_n \cdot n}{g_1^n h_1^{n+1}}$

$g_1 = \dfrac{g' H G_n}{g_1^n h_1^{n+1}}$

$(g_1 h_1)^{n+1} = g' H G_n$

$(g_1 h_1)^n (g_1 h_1) = g' H G_n$

$H_n G_n = h_1^n g_1^n$ (note)

$(H_n G_n)(g_1 h_1) = g' H G_n$

$H = H_n h'$ (note)

$(H_n G_n)(g_1 h_1) = g' (H_n G_n) h'$

$g_1 h_1 = g' h'$ $\square$